not be zero. In this case, (1) must be corrected, since the plunger measures the integral $\int(\epsilon E^2 - \mu H^2)dV$ rather than the desired $E_0^2\Delta V$. This can be taken into account by the correction factor α which is approximately given by $\alpha = (\frac{3}{8})\ (ka)^2$, assuming the fields vary in a manner similar to the principal mode in a cylindrical cavity. Here a is the plunger radius and $k_1 = 2.4/\rho_2$. Results for a particular case are shown in Fig. 7 (previous page), indicating that the correction is fair for the particular configuration. The accuracy of this correction for other configurations is a matter of some conjecture. In general, for the best results, the perturbing volume should be as small as possible.

The effect of coupling-loop size and contact pressure on R/Q measurements was also investigated, and as expected, did not affect the value of R/Q, but shifted the resonant frequency slightly.

Conclusions

The values of R/Q of klystron cavities obtained by using radial-field or coaxial-field distributions are very useful, except for a class of configurations approaching "square" resonators. For this class of cavities, the mean value of the two field approximations gives values of R/Q which are in general too low. The error, as determined by experiment and net point calculations, does not exceed 15 per cent, for values of $k\rho_1 \leqq 0.6$ (provided $kd < 0.5$) and for values of $k\rho_1 \leqq 0.8$ (provided $kd < 0.3$). Accurate values of R/Q for these configurations are also given. For cavities which do not fall within above specifications, the fields should be determined by net point methods, or by experiment. Effects of perturbing plunger sizes on R/Q measurements may also be important if plunger diameter is comparable to post diameter; a correction factor should then be applied.

Planar Transmission Lines—II*

DAVID PARK†

Summary—**An expression is found for the characteristic impedance of a transmission line consisting of two parallel strips of foil placed between, and perpendicular to, two wide plates.**

Introduction

CONTINUING the investigation of an earlier paper,[1] of transmission lines composed of flat strips of metal or foil, we examine here the characteristics of a line in which the strips no longer lie in the same plane. We shall be concerned with a configuration, shown in Fig. 1(a), in which the two center strips are perpendicular to the top and bottom sheets, midway between them, and separated from each other by distance $2D$. The center strips are each of height $2C$, and the separation between the top and bottom sheets is $2H$. The center strips are driven, and the top and bottom sheets are considered to be electrically neutral and effectively infinite in width.[2] We shall use the notations and, as far as possible, the results of the earlier paper to find, by the method of conformal mapping, the characteristic impedance Z_0 of the line in terms of H, C, D, and the dielectric constant κ of the dielectric material between the plates. (To calculate the attenuation by the methods of the earlier paper is straightforward though rather onerous, and we have not carried it out.)

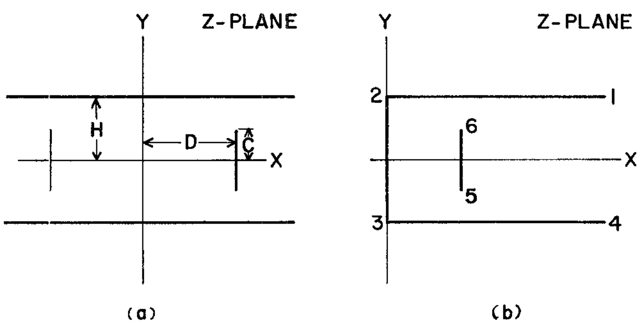

Fig. 1—(a) Cross section of the transmission line. The vertical strips are driven and the horizontal plates, assumed very wide, are electrically neutral. (b) Arrangement electrically equivalent to (a). Some significant points are numbered.

General Formalism

To begin with, let us substitute for the arrangement of Fig. 1(a) that of Fig. 1(b), in which the left-hand side of the line is substituted by its image in the vertical center plane. As mentioned,[3] the line 1234 can be mapped by the transformation

$$z = A \sinh Z/B \tag{1}$$

into the y axis of the z plane, if

$$(1/2)\pi B = H. \tag{2}$$

We must now see what this mapping does to the internal

* Supported by Sprague Electric Co., North Adams, Mass.
† Williams College, Williamstown, Mass. Fullbright Lecture in Physics at the Univ. of Ceylon, 1955–6).
[1] D. Park, "Planar transmission lines," Trans. IRE, vol. MTT-3, pp. 8–12; April, 1955.

[2] *Ibid.*, section 5, for an examination of this assumption in the case dissussed there.
[3] *Ibid.*

strip. Setting $Z=D+jY$ in (1), we find that

$$\frac{z}{A}=\frac{x+jy}{A}=\sinh\frac{D}{B}\cos\frac{Y}{B}+j\cosh\frac{D}{B}\sin\frac{Y}{B}$$

whence, equating real and imaginary parts and eliminating Y, we find that the image curve of the strip is the ellipse

$$\frac{x^2}{A^2\sinh^2 D/B}+\frac{y^2}{A^2\cosh^2 D/B}=1 \qquad (3)$$

as shown in Fig. 2(a) where α and β, the major and minor axes of the ellipse, are $A\cosh D/B$ and $A\sinh D/B$, respectively. This arrangement is of course equivalent to that of Fig. 2(b), and at this point one can see one's way to a solution, for if the ellipse is mapped onto a circle, this circle can, by a suitable linear fractional transformation, be mapped onto a straight line. The two strips will then be represented by two segments of this line, and the problem can be completed.[4]

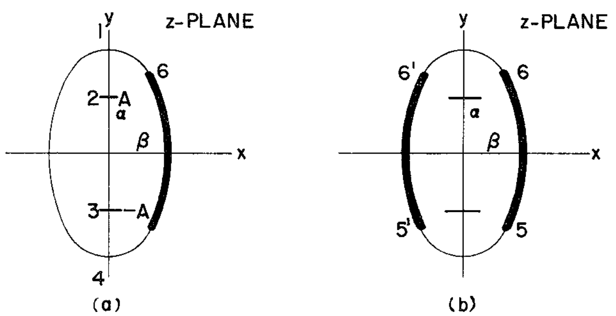

Fig. 2—(a) The line 1234 of Fig. 1(b) has been mapped into the y-axis, and the strip 56 into a segment of an ellipse. (b) Arrangement electrically equivalent to (a).

The mapping which carries the ellipse in the z plane into a circle of radius ρ in the z' plane is[5]

$$jz'=\sqrt{k\rho}\;\mathrm{sn}\left[\frac{2}{\pi}jK\sinh^{-1}\frac{z}{\sqrt{(\alpha^2-\beta^2)}}\right] \qquad (4)$$

where sn x is the Jacobian elliptic function of modulus k, K is the complete elliptic integral of the first kind (and of modulus k), and k is defined in terms of the modular function q by the relation[6,7]

$$q(k)=e^{-\pi K'/K}=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2. \qquad (5)$$

If we put in the values of α and β and then use (1) to express z in terms of Z, we get

$$jz'=\sqrt{k\rho}\;\mathrm{sn}\left(\frac{2jKZ}{\pi B}\right)\quad(q=e^{-4D/B}). \qquad (6)$$

[4] *Ibid.*

[5] This is derived by a rotation of axes from a mapping given by H. Kober, "Dictionary of Conformal Representations," Dover Publications, Inc., New York, N. Y., p. 177; 1952.

[6] Park, *op. cit.*

[7] See E. B. Wilson, "Advanced Calculus," Ginn and Co., Boston, Mass., chaps. 17 and 19; 1911.

This mapping is shown in Fig. 3.

To complete the transformation, let us approach the configuration of Fig. 3 from the other direction, starting

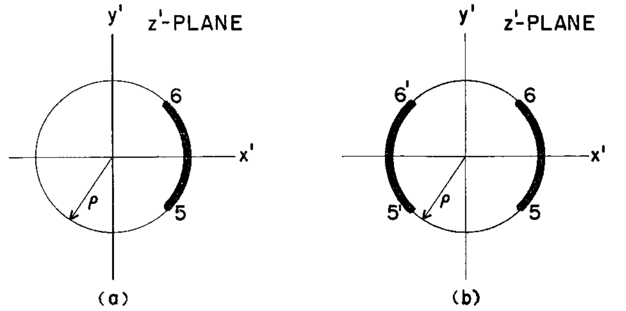

Fig. 3—(a) The ellipse of Fig. 2(a) has been mapped into a circle of radius ρ. (b) Arrangement electrically equivalent to (a).

with that of Fig. 4, for which a solution has already been found.[1] The general transformation, mapping circles in one plane into straight lines in another, is the linear-fractional one, which has the form $z'''=(pz''+q)/(rz''+s)$. To determine the constants we require that the image points corresponding to the edges of the strips in the z'' plane lie respectively horizontally and vertically opposite each other, as in Fig. 3(b). The calculation is very straightforward, and yields

$$z'''=\frac{-\sqrt{(b/a)}z''+ja}{jz''/d-2c/w}\quad(c=b-a) \qquad (7)$$

where c represents the width of the flat strips and d and w are limited by the relation

$$ab=\left(\frac{cd}{2w}\right)^2. \qquad (8)$$

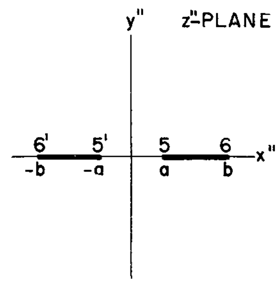

Fig. 4—Two-strip transmission line which is electrically equivalent to Fig. 3(b) and for which a solution has already been found.[1]

The real axis of Fig. 4 is mapped into the circle of Fig. 5, whose center is easily found to be at a height w above the origin. Further, it is seen from Fig. 5 that if ρ is the radius of the circle, then

$$\rho^2=d^2+w^2. \qquad (9)$$

Finally, we can express z''' in terms of z' by the relation

$$z'''=z'-jw \qquad (10)$$

or, using (6) and (7),

$$\frac{\sqrt{(b/a)}z'' - ja}{-jz''/d + 2c/w} = j\left[w + \sqrt{k\rho}\,\mathrm{sn}\left(\frac{2jKZ}{\pi B}\right)\right]. \quad (11)$$

Now we must find out how d and w, and after that a and b, depend on the numbers C, D and H which define the original configuration. For this it is convenient to return to (10), and note that the point 5 is at $(d, 0)$ in the z''' plane and at $(D, -jC)$ in the Z plane. Equating these, and using (6), we have

$$jd = \sqrt{k\rho}\,\mathrm{sn}\left[\frac{2K}{\pi B}(C + jD)\right] + w$$

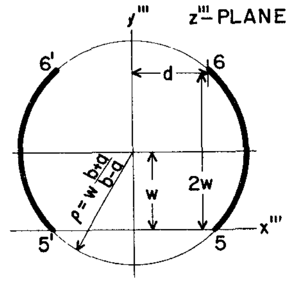

Fig. 5—The real axis of Fig. 4 has been mapped into a circle corresponding, except for the position of the origin, to that of Fig. 3(b).

from which, by comparing real and imaginary parts, we get

$$d = \rho\frac{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{1 + k\,\mathrm{sn}^2\mu}, \qquad w = (k + 1)\rho\frac{\mathrm{sn}\,\mu}{1 + k\,\mathrm{sn}^2\mu} \quad (12)$$

where

$$\mu = \frac{2KC}{\pi B} = \frac{KC}{H} = \frac{K'C}{2D} \quad (13)$$

by (2), (5) and (6), and as a check one can verify that d and w satisfy (9).

These results enable us to write down the line's characteristic impedance Z_0[8] for it is shown that the (final) mapping

$$z'' = a\,\mathrm{sn}\,W \quad (14)$$

maps the isolated segments of Fig. 4 onto the continuous lines of Fig. 6, so that the continuous ladder of lines of force in the W plane, extending horizontally from one vertical to the other, is mapped through these successive transformations into the complex pattern which they exhibit in the Z plane. It is shown[9] that if we write l for the modulus of the elliptic function in (14), define l' by

$$l' = \sqrt{(1 - l^2)}, \quad (15)$$

and write L and L' for the complete elliptic integrals of the first kind associated with the moduli l and l', then l is determined by the relation

$$b = a/l \quad (16)$$

and that in terms of it

$$Z_0 = \sqrt{\left(\frac{\mu}{\epsilon}\right)}\frac{L}{L'} \quad (17)$$

where μ and ϵ are the permeability and permittivity of the dielectric material.

In order to find l we turn back to (8) which, remembering that c is $b-a$, gives

$$\frac{1}{l} = \frac{d^2}{4w^2}\left(\frac{1}{l} - 1\right)^2,$$

or, choosing the root that puts l between 0 and 1,

$$l = 1 - 2\frac{w^2}{d^2}\left[\sqrt{\left(1 + \frac{d^2}{w^2}\right)} - 1\right], \quad (18)$$

for this has the limiting forms

$$l \approx \begin{cases} \dfrac{d^2}{4w^2}\left(1 - \dfrac{d^2}{2w^2}\right) & (d \ll w) \\ 1 - 2\dfrac{w}{d} + 2\dfrac{w^2}{d^2} & (d \gg w). \end{cases} \quad (19)$$

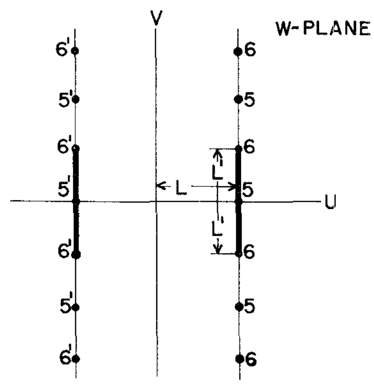

Fig. 6—The darkened segments of the real axis of Fig. 4 have been mapped into two vertical lines. The segments darkened here correspond to a complete circuit, top and bottom, of those of Fig. 4. Above and below these segments the mapping repeats itself.

Putting (12) into (18), we find that l can be expressed in two forms:

$$\sqrt{l} = \frac{(1 - k\,\mathrm{sn}\,\mu)(1 - \mathrm{sn}\,\mu)}{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu} \quad (20a)$$

or

$$l = \frac{(1 - \mathrm{sn}\,\mu)(1 - k\,\mathrm{sn}\,\mu)}{(1 + \mathrm{sn}\,\mu)(1 + k\,\mathrm{sn}\,\mu)}. \quad (20b)$$

In order to find Z_0, we need only write [10]

$$\frac{L'}{L} = \frac{1}{\pi}f(l^2) = \frac{\pi}{f(l'^2)} \quad (21)$$

[8] Park, *op. cit.*
[9] *Ibid.*
[10] *Ibid.*

where f is the functional plotted in Fig. 7. Then, if the permeability of the dielectric medium is taken to be that of free space, so that $\sqrt{(\mu/\epsilon)}=120\pi/\sqrt{\kappa}$, where κ is the medium's dielectric constant, we have

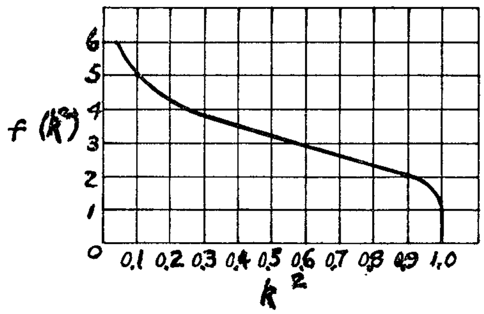

Fig. 7—Plot of $f(k^2)=(\ln)1/q$ against k^2. Note that $f(k^2)=\pi^2/f(k'^2)$, where $k^2+k'^2=1$.

$$Z_0=\frac{120f(l'^2)}{\sqrt{\kappa}}=\frac{120\pi^2}{\sqrt{\kappa}f(l^2)}\cdot \tag{22}$$

It is now a straightforward matter to carry out the calculation of Z_0 for any line. First it is necessary to find k, defined by (6). We can use (2) to write

$$\ln\frac{1}{q}=2\pi\frac{D}{H}$$

and the quantity on the left, designated as $f(k^2)$, is plotted against k^2 in Fig. 7. Next, one substitutes k and μ, which is defined in terms of k by (13), into (20) to find l. With l, one can use Fig. 7 again to find Z_0 via (22). But this process, though a simple enough in principle, is cumbersome in practice and suffers from the further disadvantage that it deprives one of any analytic insight into the nature of the relationships involved. It is therefore advisable to look into ways in which the elliptic functions used here can be approximated by functions of more familiar types and the result written as a single formula for Z_0 in terms of the parameters describing the line. We have found a number of ways in which this can be done, corresponding to various inequalities which the parameters may satisfy. In the following section we shall examine approximations which are valid when the center strips reach nearly to the top and and bottom plates, so that the field between strip and plate is much stronger than that between strips. By way of example, we shall carry this out in detail, first for a line in which the separation between the strips is large compared with that between the plates, since this can be done by elementary methods; and then with this restriction removed, for which it is necessary to make use of some more profound properties of elliptic functions.

Approximations Valid when $H-C\ll H$

The Case $D\gg H$

To illustrate how the work of computation can be done approximately, let us consider the case in which the separation between the strips is large compared with the distance between the top and bottom plates: $D\gg H$. This means that q in (6) is small. To find k we can use the relation[11]

$$16q=k^2+(1/2)k^4+\frac{21}{64}k^6+\cdots.$$

to obtain

$$k^2=(2e^{-D/B})^4-\frac{1}{2}(2e^{-D/B})^8+\cdots. \tag{23}$$

Further, since sn $2Kx/\pi$ differs from sin x only by terms in k^2, we have from (20b), to the first power in k,

$$l\approx\frac{1-\sin\frac{\pi}{2}\frac{C}{H}}{1+\sin\frac{\pi}{2}\frac{C}{H}}\left(1-8e^{-\pi D/H}\sin\frac{\pi}{2}\frac{C}{H}\right). \tag{24}$$

Suppose now that C is nearly H, and write

$$\frac{C}{H}=1-\epsilon \qquad (\epsilon\ll 1). \tag{25}$$

Then

$$l\approx\frac{\pi^2\epsilon^2}{16}(1-8e^{-\pi D/H}). \tag{26}$$

Using the approximation[1]

$$f(l^2)\approx 2\ln\frac{4}{l} \qquad (l\ll 1), \tag{27}$$

we find that in this case

$$Z_0\approx\frac{30\pi^2/\sqrt{K}}{\ln\frac{8H}{\pi(H-C)}+4e^{-\pi D/H}} \qquad (H-C\ll H\ll D). \tag{28}$$

From this we can at once obtain the impedance of a line with only one center strip by letting D become infinite and dividing Z_0 by 2 because the potential is now applied between the strip and the outer plates:

$$Z_0\approx\frac{15\pi^2}{\sqrt{K}\ln\frac{8H}{\pi(H-C)}} \qquad (H-C\ll H,\ \text{single strip}). \tag{29}$$

The General Case

The situation when D is not large compared with H can be treated somewhat less easily; for variety we shall start from (19). Writing as before (by (13)) $\mu=K(1-\epsilon)$, we note that since cn $K=0$, we have $d/w\approx 0$. Use of the addition theorems for the Jacobian elliptic functions gives

$$\frac{d}{w}=\frac{k'}{1+k}\frac{\text{sn } K\epsilon}{\text{dn } K\epsilon} \tag{30}$$

[11] *Ibid.*

so that[12]

$$l \approx \frac{1-k}{1+k}\frac{K^2\epsilon^2}{4} \qquad (\epsilon^2 \ll 1). \tag{31}$$

This can be reduced to a series in q which converges well when q is not too close to unity. It follows from known formulas in elliptic functions and their transformations[13] that

$$\ln K = \ln\frac{\pi}{2} + 4\left(\frac{q}{1+q} + \frac{1}{3}\frac{q^3}{1+q^3} + \frac{1}{5}\frac{q^5}{1+q^5} + \cdots\right) \tag{32}$$

and

$$\ln\frac{1-k}{1+k} = -8\sqrt{q}\left(\frac{1}{1-q} + \frac{1}{3}\frac{q}{1-q^3} + \frac{1}{5}\frac{q^2}{1-q^5} + \cdots\right) \tag{33}$$

so that if we write $q=p^2$,

$$\ln\left(\frac{1-k}{1+k}K^2\right) = 2\ln\frac{\pi}{2} - 8\left(\frac{p-p^2+p^3+p^4}{1-p^4} + \frac{1}{3}\frac{p^3-p^6+p^9+p^{12}}{1-p^{12}} + \cdots\right). \tag{34}$$

Thus, expanding a few terms,

$$\ln l = 2\ln\frac{\pi\epsilon}{4} - 8\left(p - p^2 + \frac{4}{3}p^3 + p^4 + \cdots\right), \tag{35}$$

whence

$$l = \frac{\pi^2\epsilon^2}{16}\left(1 - p + \frac{3}{2}p^2 + \frac{5}{2}p^3 + \frac{11}{8}p^4 + \cdots\right)^8 \tag{36}$$

of which the first term again gives (26). The use of (27) and (35) now gives us

$$Z_0 = \frac{30\pi^2/\sqrt{K}}{\ln\frac{8H}{\pi(H-C)} + 4\left(p - p^2 + \frac{4}{3}p^3 + p^4 + \cdots\right)} \qquad (H - C \ll H) \tag{37}$$

with

$$p = \sqrt{q} = e^{-2D/B} = e^{-\pi D/H}$$

As long as $H-C$ remains less than, say, $H/4$, this is an excellent approximation for any configuration that one would be likely to use in practice.

I should like to express my thanks to Dr. A. Morrison for his helpful suggestions in the preparation of this paper.

[12] Note that higher terms in this expansion are easy to obtain, since (30) is exact.

[13] C. G. J. Jacobi, "Werke," Berlin, vol. 1, pp. 148, 159; 1881.

A High-Speed Broadband Microwave Waveguide Switch*

W. L. TEETER†

Summary—**A switch which switches microwave energy to any of several separate waveguide loads is described. The switch has the bandwidth and power-carrying capability which is essentially that of the input and output waveguides. Data is given for a switch which operates over the frequency range of 8,600 to 10,000 mc with a vswr of less than 1.15 during transmission and less than 1.5 during switching. The switching speed is limited only by the practical limit for rotating the metal shorting vane. A typical example is given of a 5-output switch with a switching rate of 1,800 per second (vane rotation of 3,600 rpm) and a dead time during switching of 14 per cent of total time. Dead time is a function of switch diameter and vane rotation rate and could be reduced by increasing the vane diameter or rotation rate.**

Introduction

THIS PAPER describes development of a high-speed waveguide switch capable of switching high power from one input to any number of outputs.

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Three types of switches were studied.

The first switch (Fig. 1, next page), places a number of waveguide tees in series. A movable vane, containing a rectangular hole is then moved across the top of each tee to select which load will receive the rf energy. All other loads are shorted. This switch has a 1 per cent bandwidth, excessively high vswr during switching (i.e., 20 to 1), and two rather critical manufacturing tolerances.

The second switch (Fig. 2, next page), has all output waveguides in shunt to provide a turnstile junction. Rf energy enters through the circular waveguide at the base. A cylindrical rotor with a hole in it rotates in such a way that the hole allows energy to pass to a particular load. This switch has a 3 per cent frequency bandwidth, excessively high vswr during switching and two critical manufacturing tolerances.[1]

[1] There is some discussion of the turnstile switch in MIT Rad. Lab. series, vol. 9, p. 538.